A fixed point approach to the stability of some functional equation connected with additive and quadratic mappings

Marcin Adam

Abstract. In this paper we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the functional equation connected with additive and quadratic mappings for a class of functions between a linear space and a complete β-normed space.

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1. Introduction

Let E_1 and E_2 be real linear spaces. We recall some basic definitions. A map $A\colon E_1 \to E_2$ is said to be additive iff it satisfies the Cauchy functional equation

$$A(x+y) = A(x) + A(y), \quad x, y \in E_1.$$

A map $Q\colon E_1 \to E_2$ is said to be quadratic iff it satisfies the following functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad x, y \in E_1.$$

In the theory of functional equations the problem of the stability has its origin in the following question, posed by S. Ulam [33] in 1940, concerning the stability of group homomorphisms:

Let G be a group and let G_1 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h\colon G \to G_1$ satisfies the inequality

$$d\big(h(xy), h(x)h(y)\big) < \delta \quad \text{for all } x, y \in G,$$

M. Adam
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: marcin.adam@polsl.pl

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then there exists a homomorphism $H\colon G \to G_1$ with

$$d\big(h(x), H(x)\big) < \varepsilon \quad \text{for all } x \in G?$$

In the next year, D.H. Hyers [18] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. That was the first significant breakthrough and a step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers' theorem. Th.M. Rassias in [27] (see also [2]), G.L. Forti in [14] and Z. Gajda in [16] considered the stability problem with unbounded Cauchy differences. The above results can be partially summarized in the following theorem:

Theorem 1.1. *Let X and Y be a real normed space and a real Banach space, respectively, and let $p \neq 1$ be a nonnegative constant. Suppose that a function $f\colon X \to Y$ satisfies the inequality*

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \varepsilon\big(\|x\|^p + \|y\|^p\big), \quad x, y \in X$$

for some $\varepsilon > 0$. Then there exists a unique additive function $A\colon X \to Y$ such that

$$\big\| f(x) - A(x) \big\| \leqslant \frac{2\varepsilon}{|2 - 2^p|} \|x\|^p, \quad x \in X.$$

This phenomenon is called Hyers-Ulam-Rassias stability. The function $A\colon X \to Y$ can be explicitly constructed, started from the given function f, by the formulae

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \quad p < 1, \quad \text{and} \quad A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad p > 1.$$

This method is called the direct method or Hyers' method. It is often used to construct a solution of a given functional equation and is a powerful tool for studying the stability of many functional equations.

The second most often considered equation is the quadratic functional equation. The Hyers-Ulam stability of this equation was first proved by F. Skof [30] and generalized by P.W. Cholewa [5]. Thereafter, S. Czerwik [8] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation and his result reads as follows:

Theorem 1.2. *Let X and Y be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a nonnegative constant. If a function $f\colon X \to Y$ satisfies the inequality*

$$\big\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \big\| \leqslant \varepsilon\big(\|x\|^p + \|y\|^p\big), \quad x, y \in X$$

for some $\varepsilon > 0$, then there exists a unique quadratic function $Q\colon X \to Y$ such that

$$\big\| f(x) - Q(x) \big\| \leqslant \frac{2\varepsilon}{|4 - 2^p|} \|x\|^p, \quad x \in X.$$

The stability problems of several functional equations have been extensively investigated by many authors. For more information and primary references, the reader should refer to the monographs [10, 11, 19, 22], and papers, e.g. [9, 15, 20, 28].

The Hyers' method is the most popular technique of proving the Hyers-Ulam stability of functional equations. Nevertheless, there are also known several different approaches proving the Hyers-Ulam stability, for example the method of invariant means (see [17, 31]), the method based on sandwich theorems (see [25]) and on the concept of shadowing (see [32]).

L. Cădariu and V. Radu applied the fixed point method to the investigation of Jensen and Cauchy functional equations (see [3] and [4], respectively). Now it is the second most popular technique of proving the Hyers-Ulam stability of functional equations. An extensive source of information on applications of fixed point theorems to the Hyers-Ulam stability of functional equations is Ciepliński's survey paper [6].

In this paper, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the following functional equation

$$f(x+3y)+3f(x-y)=f(x-3y)+3f(x+y), \tag{1}$$

which is connected with additive and quadratic mappings. We will consider a class of functions between a linear space and a complete β-normed space. The above functional equation is interesting because of its connection with a single variable functional equation (see [1, 13, 29]) which is also closely associated with additive and quadratic mappings. It is well known (see [21]) that the general solution of (1) in the class of functions between real or complex linear spaces is of the form $f = Q + A + c$, where Q is a quadratic mapping, A is an additive one and $c = f(0)$.

It is worth to notice that the equation (1) is equivalent to the functional equation $\Delta^3_{2y} f(x-3y) = 0$, where Δ is the difference operator defined by $\Delta_h f(x) = f(x+h) - f(x)$ and Δ^3 denotes its third iterate. Therefore a solution of the above equation is a polynomial of degree at most two (see, e.g., [24]).

Standard symbols $\mathbb{R}$, $\mathbb{C}$ denote the sets of real and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of positive integers.

2. Preliminaries

In this section we present some definitions and auxiliary result which will be needed in the sequel.

Definition 2.1 ([12]). *Let X be a nonempty set. A function $d\colon X \times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies the following conditions:*

(i) $d(x,y) = 0$ *if and only if* $x = y$,
(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$,
(iii) $d(x,z) \leqslant d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [12].

Theorem 2.2 ([12])**.** *Let (X, d) be a generalized metric space. Assume that $\Lambda\colon X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either*

(a) $d(\Lambda^{n+1}x, \Lambda^n x) = \infty$ for all $n \in \mathbb{N}_0$,

or

(b) there exists $k \in \mathbb{N}_0$ such that $d(\Lambda^{n+1}x, \Lambda^n x) < \infty$ for all $n \geqslant k$, $n \in \mathbb{N}_0$.

Actually, if (b) holds and the respective $k \in \mathbb{N}_0$ is fixed, then

(i) the sequence $(\Lambda^n x)_{n\in\mathbb{N}_0}$ converges to the fixed point x^* of Λ,
(ii) x^* is the unique fixed point of Λ in the space

$$X^* := \{y \in X : d(\Lambda^k x, y) < \infty\},$$

(iii) if $y \in X^*$, then

$$d(y, x^*) \leqslant \frac{1}{1-L} d(\Lambda y, y).$$

Remark 2.3. *It is known that the fixed point x^*, if it exists, is not necessarily unique in the whole space X, it may depend on the starting point x. Moreover, in the case (b), the pair (X^*, d) is a complete metric space and $\Lambda(X^*) \subset X^*$. Therefore the properties (i)–(iii) follow from Banach's Contraction Principle (cf. [23]).*

Let E be a vector space over the field $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$. Moreover, from now, let $\alpha \in (0, \infty)$ and $0 < \beta \leqslant 1$.

Definition 2.4 (cf. [4])**.** *A mapping $\|\cdot\|_\alpha\colon E \to [0, \infty)$ is called a sub-homogeneous functional of order α if and only if*

$$\|\lambda x\|_\alpha \leqslant |\lambda|^\alpha \cdot \|x\|_\alpha, \quad \lambda \in \mathbb{K},\ x \in E. \tag{2}$$

Similarly, we can formulate the following definition.

Definition 2.5. *A mapping $\|\cdot\|_\alpha\colon E \to [0, \infty)$ is called a sub-homogeneous functional of order 2α if and only if*

$$\|\lambda x\|_\alpha \leqslant |\lambda|^{2\alpha} \cdot \|x\|_\alpha, \quad \lambda \in \mathbb{K},\ x \in E. \tag{3}$$

Actually, a sub-homogeneous functional of order α (or 2α) is a homogeneous functional of order α (or 2α). Indeed, it suffices to substitute λx and $\frac{1}{\lambda}$ in the place of x and λ in the above conditions, respectively, to obtain the converse inequalities.

As usually, E is identified with $E \times \{0\}$ in $E \times E$. Hence $\|x\|_\alpha = \|(x, 0)\|_\alpha$ for all $x \in E$ and for each sub-homogeneous functional of order α (or 2α) on $E \times E$.

Definition 2.6 ([4])**.** *A mapping $\|\cdot\|_\beta\colon E \to [0, \infty)$ is called a β-norm if and only if it has the following properties:*

(i) $\|x\|_\beta = 0$ if and only if $x = 0$,
(ii) $\|\lambda x\|_\beta = |\lambda|^\beta \cdot \|x\|_\beta$,
(iii) $\|x + y\|_\beta \leqslant \|x\|_\beta + \|y\|_\beta$

for all $\lambda \in \mathbb{K}$ and $x, y \in E$.

3. Main results

Throughout this section let E_1, E_2 be two linear spaces over the same field $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$. Moreover, assume that E_2 is a complete β-normed space for some $0 < \beta \leqslant 1$. Let us define a number a_i, $i = 0, 1$, by the formula:

$$a_i = \begin{cases} 2, & i = 0, \\ \frac{1}{2}, & i = 1. \end{cases}$$

To shorten some considerations and for the sake of brevity we shall use in the following two theorems the same notations for the case $i = 0$ and $i = 1$. We investigate the stability problem of (1) by decomposing the unknown function f into its even and odd part. Next, we will connect these two cases receiving the main Theorem 3.3.

Theorem 3.1. *Suppose $\varphi\colon E_1 \times E_1 \to [0, \infty)$ is a given function and there exists a constant L, $0 < L < 1$, such that the mapping*

$$x \to \psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \quad x \in E_1,$$

has the property

$$\psi(x) \leqslant L \cdot a_i^{2\beta} \psi\left(\frac{x}{a_i}\right), \quad x \in E_1,\ i = 0, 1, \tag{4}$$

and the mapping φ satisfies the condition

$$\lim_{n\to\infty} \frac{\varphi(a_i^n x, a_i^n y)}{a_i^{2n\beta}} = 0, \quad x, y \in E_1,\ i = 0, 1. \tag{5}$$

If an even function $f\colon E_1 \to E_2$ satisfies the inequality

$$\left\|f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y)\right\|_\beta \leqslant \varphi(x, y), \quad x, y \in E_1,$$

then there exists a unique quadratic mapping $Q\colon E_1 \to E_2$ such that

$$\left\|f(x) - f(0) - Q(x)\right\|_\beta \leqslant \frac{1}{4^\beta} \frac{L^i}{1-L} \psi(x), \quad x \in E_1,\ i = 0, 1. \tag{6}$$

Proof. Let $f_1(x) := f(x) - f(0)$ for all $x \in E_1$. Then $f_1(0) = 0$ and the function f_1 satisfies the inequality

$$\left\|f_1(x+3y) + 3f_1(x-y) - f_1(x-3y) - 3f_1(x+y)\right\|_\beta \leqslant \varphi(x, y), \quad x, y \in E_1. \tag{7}$$

Consider the set

$$X := \left\{h\colon E_1 \to E_2 : h(0) = 0\right\},$$

and introduce the generalized metric on X:

$$d(g, h) := \inf\left\{C \in [0, \infty] : \left\|g(x) - h(x)\right\|_\beta \leqslant C\psi(x)\right\}, \quad x \in E_1.$$

As usually, $\inf \emptyset := \infty$. First, we will verify that (X, d) is a complete space. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d), i.e.

$$\bigwedge_{\varepsilon>0} \bigvee_{N_0\in\mathbb{N}} \bigwedge_{m,n\geqslant N_0} d(g_m, g_n) \leqslant \varepsilon.$$

By considering the definition of the generalized metric d, we see that

$$\bigwedge_{\varepsilon>0} \bigvee_{N_0\in\mathbb{N}} \bigwedge_{m,n\geqslant N_0} \bigwedge_{x\in E_1} \big\|g_m(x) - g_n(x)\big\|_\beta \leqslant \varepsilon\psi(x). \tag{8}$$

For fixed $x \in E_1$, condition (8) implies that $\big(g_n(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(E_2, \|\cdot\|_\beta\big)$. Since $\big(E_2, \|\cdot\|_\beta\big)$ is complete, $\big(g_n(x)\big)_{n\in\mathbb{N}}$ converges in $\big(E_2, \|\cdot\|_\beta\big)$ for each $x \in E_1$. Hence we can define a function $g\colon E_1 \to E_2$ by

$$g(x) := \lim_{n\to\infty} g_n(x), \quad x \in E_1.$$

Letting $m \to \infty$ in (8), we get

$$\bigwedge_{\varepsilon>0} \bigvee_{N_0\in\mathbb{N}} \bigwedge_{n\geqslant N_0} \bigwedge_{x\in E_1} \big\|g(x) - g_n(x)\big\|_\beta \leqslant \varepsilon\psi(x),$$

i.e.

$$\bigwedge_{\varepsilon>0} \bigvee_{N_0\in\mathbb{N}} \bigwedge_{n\geqslant N_0} d(g, g_n) \leqslant \varepsilon.$$

This fact leads us to the conclusion that the sequence $(g_n)_{n\in\mathbb{N}}$ converges in (X, d). Hence (X, d) is a complete space.

We define an operator $\Lambda\colon X \to X$ by the formula

$$(\Lambda h)(x) := \frac{1}{a_i^2} h(a_i x), \quad x \in E_1,\ i = 0, 1.$$

We show that Λ is strictly contractive on X. Given $g, h \in X$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leqslant C$, i.e.

$$\big\|g(x) - h(x)\big\|_\beta \leqslant C\psi(x), \quad x \in E_1,\ i = 0, 1.$$

Substituting $a_i x$ instead of x in the above inequality and dividing both sides of the resulting expression by $a_i^{2\beta}$, we get

$$\left\|\frac{g(a_i x)}{a_i^2} - \frac{h(a_i x)}{a_i^2}\right\|_\beta \leqslant \frac{1}{a_i^{2\beta}} C\psi(a_i x), \quad x \in E_1,\ i = 0, 1.$$

Therefore in view of (4) and the definition of Λ we see that

$$\big\|(\Lambda g)(x) - (\Lambda h)(x)\big\|_\beta \leqslant LC\psi(x), \quad x \in E_1,$$

i.e. $d(\Lambda g, \Lambda h) \leqslant LC$. Hence we conclude that $d(\Lambda g, \Lambda h) \leqslant Ld(g, h)$ for any $g, h \in X$.

Next, we assert that $d(\Lambda f_1, f_1) < \infty$. Consider the case where $i = 0$. Substituting $\frac{x}{2}$ instead of x and y in (7) and dividing both sides of the resulting inequality by 4^β, we obtain

$$\big\|(\Lambda f_1)(x) - f_1(x)\big\|_\beta \leqslant \frac{1}{4^\beta}\psi(x), \quad x \in E_1,$$

i.e.

$$d(\Lambda f_1, f_1) \leqslant \frac{1}{4^\beta} < \infty.$$

Let us now consider the second case where $i = 1$. Replacing x and y by $\frac{x}{4}$ in (7) and applying (4) gives

$$\big\|(\Lambda f_1)(x) - f_1(x)\big\|_\beta \leqslant \frac{1}{4^\beta}L\psi(x), \quad x \in E_1,$$

i.e.

$$d(\Lambda f_1, f_1) \leqslant \frac{1}{4^\beta}L < \infty.$$

Therefore we have

$$d(\Lambda f_1, f_1) \leqslant \frac{1}{4^\beta}L^i < \infty, \quad i = 0, 1. \tag{9}$$

By Theorem 2.2 (i) there exists a mapping $Q\colon E_1 \to E_2$ with $Q(0) = 0$, which is a fixed point of Λ, i.e. $(\Lambda Q)(x) = Q(x)$ for all $x \in E_1$. Hence $Q(2x) = 4Q(x)$ for all $x \in E_1$ and $\Lambda^n f_1 \to Q$, i.e.

$$\lim_{n\to\infty} \frac{1}{a_i^{2n}} f_1(a_i^n x) = Q(x), \quad x \in E_1,\ i = 0, 1.$$

Since $k = 0$ (see (9)) and $f_1 \in X^*$ in Theorem 2.2, by Theorem 2.2 (iii) and (9) we obtain

$$d(f_1, Q) \leqslant \frac{1}{1-L} d(\Lambda f_1, f_1) \leqslant \frac{1}{4^\beta}\frac{L^i}{1-L},$$

i.e.

$$\big\|f_1(x) - Q(x)\big\|_\beta \leqslant \frac{1}{4^\beta}\frac{L^i}{1-L}\psi(x), \quad x \in E_1,\ i = 0, 1,$$

which means that the inequality (6) is true.

We verify that a function Q is quadratic. Substituting $a_i^n x$ and $a_i^n y$ instead of x and y in (7), respectively, and dividing both sides of the resulting inequality by $a_i^{2n\beta}$, we get

$$\left\|\frac{f_1\big(a_i^n(x+3y)\big)}{a_i^{2n}} + \frac{3f_1\big(a_i^n(x-y)\big)}{a_i^{2n}} - \frac{f_1\big(a_i^n(x-3y)\big)}{a_i^{2n}} - \frac{3f_1\big(a_i^n(x+y)\big)}{a_i^{2n}}\right\|_\beta \leqslant \frac{\varphi(a_i^n x, a_i^n y)}{a_i^{2n\beta}}, \quad x, y \in E_1,$$

where $i = 0, 1$. Taking the limit in the above expression as $n \to \infty$ and applying (5) we conclude that

$$Q(x+3y)+3Q(x-y)=Q(x-3y)+3Q(x+y), \quad x,y\in E_1.$$

This implies that Q is a quadratic function (since Q is even, cf. [21]).

To prove the uniqueness of the solution assume that there exists another quadratic function $Q_1\colon E_1\to E_2$ satisfying the condition (6). Therefore $Q_1(x)=\frac{1}{a_i^2}Q_1(a_ix)=(\Lambda Q_1)(x)$ for all $x\in E_1$, i.e. Q_1 is a fixed point of Λ. In view of (6) with Q_1 and the definition of d, we know that

$$\big\|f_1(x)-Q_1(x)\big\|_\beta \leqslant \frac{1}{4^\beta}\frac{L^i}{1-L}\psi(x), \quad x\in E_1,\ i=0,1,$$

i.e.

$$d(f_1,Q_1)\leqslant \frac{1}{4^\beta}\frac{L^i}{1-L}<\infty,\ i=0,1.$$

Thus $Q_1\in X^*=\big\{y\in X\colon d(\Lambda f_1,y)<\infty\big\}$ and Theorem 2.2 (ii) implies that $Q=Q_1$, which proves the uniqueness of Q. The proof is completed. □

Theorem 3.2. *Suppose $\varphi\colon E_1\times E_1\to[0,\infty)$ is a given function and there exists a constant L, $0<L<1$, such that the mapping*

$$x\to\psi(x)=\varphi\left(\frac{x}{2},\frac{x}{2}\right), \quad x\in E_1$$

has the property

$$\psi(x)\leqslant L\cdot a_i^\beta\psi\left(\frac{x}{a_i}\right), \quad x\in E_1,\ i=0,1, \tag{10}$$

and the mapping φ satisfies the condition

$$\lim_{n\to\infty}\frac{\varphi(a_i^nx,a_i^ny)}{a_i^{n\beta}}=0, \quad x,y\in E_1,\ i=0,1. \tag{11}$$

If an odd function $f\colon E_1\to E_2$ satisfies the inequality

$$\big\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\big\|_\beta\leqslant\varphi(x,y), \quad x,y\in E_1, \tag{12}$$

then there exists a unique additive mapping $A\colon E_1\to E_2$ such that

$$\big\|f(x)-A(x)\big\|_\beta\leqslant\frac{1}{2^\beta}\frac{L^i}{1-L}\psi(x), \quad x\in E_1,\ i=0,1. \tag{13}$$

Proof. Since a function f is odd then obviously $f(0)=0$. Similarly as in the proof of Theorem 3.1 we define the set X and the generalized metric d. Then (X,d) is a complete space.

We define an operator $\Lambda\colon X\to X$ by the formula

$$(\Lambda h)(x):=\frac{1}{a_i}h(a_ix), \quad x\in E_1,\ i=0,1.$$

We assert that Λ is strictly contractive on X. For given $g, h \in X$, let $C \in [0,\infty]$ be an arbitrary constant such that $d(g,h) \leqslant C$, i.e.

$$\left\| g(x) - h(x) \right\|_\beta \leqslant C\psi(x), \quad x \in E_1.$$

Substituting $a_i x$ instead of x in the above inequality and dividing both sides of the resulting expression by a_i^β, we obtain

$$\left\| \frac{g(a_i x)}{a_i} - \frac{h(a_i x)}{a_i} \right\|_\beta \leqslant \frac{1}{a_i^\beta} C\psi(a_i x), \quad x \in E_1,\ i = 0, 1.$$

Therefore in view of (10) and the definition of Λ we see that

$$\left\| (\Lambda g)(x) - (\Lambda h)(x) \right\|_\beta \leqslant LC\psi(x), \quad x \in E_1,$$

i.e. $d(\Lambda g, \Lambda h) \leqslant LC$. Thus $d(\Lambda g, \Lambda h) \leqslant L d(g,h)$ for any $g, h \in X$.

We show that $d(\Lambda f, f) < \infty$. Consider the case where $i = 0$. Substituting $\frac{x}{2}$ instead of x and y in (12) and dividing both sides of the resulting inequality by 2^β, we obtain

$$\left\| (\Lambda f)(x) - f(x) \right\|_\beta \leqslant \frac{1}{2^\beta}\psi(x), \quad x \in E_1,$$

i.e.

$$d(\Lambda f, f) \leqslant \frac{1}{2^\beta} < \infty.$$

Let us now consider the second case where $i = 1$. Replacing x and y by $\frac{x}{4}$ in (12) and applying (10) gives

$$\left\| (\Lambda f)(x) - f(x) \right\|_\beta \leqslant \frac{1}{2^\beta} L\psi(x), \quad x \in E_1,$$

i.e.

$$d(\Lambda f, f) \leqslant \frac{1}{2^\beta} L < \infty.$$

Therefore we have

$$d(\Lambda f, f) \leqslant \frac{1}{2^\beta} L^i < \infty, \quad i = 0, 1. \tag{14}$$

Then, it follows from Theorem 2.2 (i) that there exists a function $A\colon E_1 \to E_2$ with $A(0) = 0$, which is a fixed point of Λ, i.e. $(\Lambda A)(x) = A(x)$ for all $x \in E_1$. Thus $A(2x) = 2A(x)$ for all $x \in E_1$ and $\Lambda^n f \to A$, i.e.

$$\lim_{n\to\infty} \frac{1}{a_i^n} f(a_i^n x) = A(x), \quad x \in E_1,\ i = 0, 1.$$

Since $k = 0$ (see (14)) and $f \in X^*$ in Theorem 2.2, by Theorem 2.2 (iii) and (14) we obtain

$$d(f, A) \leqslant \frac{1}{1-L} d(\Lambda f, f) \leqslant \frac{1}{2^\beta}\frac{L^i}{1-L},$$

i.e.

$$\big\|f(x) - A(x)\big\|_\beta \leqslant \frac{1}{2^\beta}\frac{L^i}{1-L}\psi(x), \quad x \in E_1,\ i = 0,1,$$

which means that the inequality (13) holds true.

We show that a function A is additive. Substituting $a_i^n x$ i $a_i^n y$ instead of x and y in (12), respectively, and dividing both sides of the resulting inequality by $a_i^{n\beta}$, we get

$$\left\|\frac{f\big(a_i^n(x+3y)\big)}{a_i^n} + \frac{3f\big(a_i^n(x-y)\big)}{a_i^n} - \frac{f\big(a_i^n(x-3y)\big)}{a_i^n} - \frac{3f\big(a_i^n(x+y)\big)}{a_i^n}\right\|_\beta \leqslant \frac{\varphi(a_i^n x, a_i^n y)}{a_i^{n\beta}}, \quad x, y \in E_1,$$

where $i = 0, 1$. Letting $n \to \infty$ in the above inequality and applying (11) we have

$$A(x+3y) + 3A(x-y) = A(x-3y) + 3A(x+y), \quad x, y \in E_1.$$

This implies that A is a additive function (since A is odd, cf. [21]).

Assume that inequality (13) is also satisfied with another additive function $A_1\colon E_1 \to E_2$ besides A. Therefore $A_1(x) = \frac{1}{a_i}A_1(a_i x) = (\Lambda A_1)(x)$ for all $x \in E_1$, i.e. A_1 is a fixed point of Λ. In view of (13) with Q_1 and the definition of d, we know that

$$\big\|f(x) - A_1(x)\big\|_\beta \leqslant \frac{1}{2^\beta}\frac{L^i}{1-L}\psi(x), \quad x \in E_1,\ i = 0,1,$$

i.e.

$$d(f, A_1) \leqslant \frac{1}{2^\beta}\frac{L^i}{1-L} < \infty,\ i = 0,1.$$

Thus $A_1 \in X^* = \big\{y \in X\colon d(\Lambda f, y) < \infty\big\}$ and Theorem 2.2 (*ii*) implies that $A = A_1$, which proves the uniqueness of A. This completes the proof. $\square$

Theorem 3.3. *Suppose $\varphi\colon E_1 \times E_1 \to [0,\infty)$ is a given function and there exists a constant L, $0 < L < 1$, such that the mapping*

$$x \to \psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \quad x \in E_1$$

has the properties:

$$\psi(x) \leqslant L \cdot a_i^{2^i\beta}\psi\left(\frac{x}{a_i}\right), \quad x \in E_1,\ i = 0,1,$$

and the mapping φ satisfies the conditions

$$\lim_{n\to\infty}\frac{\varphi(a_i^n x, a_i^n y)}{a_i^{2^i n\beta}} = 0, \quad x, y \in E_1,\ i = 0,1.$$

If a function $f\colon E_1 \to E_2$ satisfies the inequality

$$\left\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\right\|_\beta \leqslant \varphi(x,y), \quad x,y\in E_1, \tag{15}$$

then there exist a unique quadratic mapping $Q\colon E_1 \to E_2$ and a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\left\|f(x)-f(0)-Q(x)-A(x)\right\|_\beta \leqslant \frac{2^\beta+1}{8^\beta}\frac{L^i}{1-L}\left[\psi(x)+\psi(-x)\right], \quad x\in E_1,\ i=0,1. \tag{16}$$

Proof. We define functions $f_1, f_2\colon E_1 \to E_2$ by

$$f_1(x) := \frac{f(x)+f(-x)}{2}, \qquad f_2(x) := \frac{f(x)-f(-x)}{2}, \quad x\in E_1.$$

Then $f = f_1 + f_2$. Since a function f safisfies the condition (15), so the following inequalities hold true

$$\left\|f_1(x+3y)+3f_1(x-y)-f_1(x-3y)-3f_1(x+y)\right\|_\beta \leqslant \frac{1}{2^\beta}\left[\varphi(x,y)+\varphi(-x,-y)\right],$$

$$\left\|f_2(x+3y)+3f_2(x-y)-f_2(x-3y)-3f_2(x+y)\right\|_\beta \leqslant \frac{1}{2^\beta}\left[\varphi(x,y)+\varphi(-x,-y)\right]$$

for all $x,y\in E_1$. It follows from Theorems 3.1 and 3.2 that there exist a unique quadratic function $Q\colon E_1\to E_2$ and aunique additive function $A\colon E_1\to E_2$ such that

$$\left\|f_1(x)-f_1(0)-Q(x)\right\|_\beta \leqslant \frac{1}{8^\beta}\frac{L^i}{1-L}\left[\psi(x)+\psi(-x)\right],$$
$$\left\|f_2(x)-A(x)\right\|_\beta \leqslant \frac{1}{4^\beta}\frac{L^i}{1-L}\left[\psi(x)+\psi(-x)\right]$$

for all $x\in E_1,\ i=0,1$, respectively. From the above inequalities we easily obtain the condition (16). □

Corollary 3.4. *Suppose that we have given a sub-homogeneous functional of order 2α on $E_1\times E_1$, $\alpha\neq\beta$. Then for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for every even function $f\colon E_1\to E_2$ which satisfies*

$$\left\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\right\|_\beta \leqslant \delta(\varepsilon)\cdot\left\|(x,y)\right\|_\alpha, \quad x,y\in E_1,$$

there exists a unique quadratic mapping $Q\colon E_1\to E_2$ such that

$$\left\|f(x)-f(0)-Q(x)\right\|_\beta \leqslant \varepsilon\cdot\left\|(x,x)\right\|_\alpha, \quad x\in E_1. \tag{17}$$

Proof. Define

$$\varphi(x,y) := \delta(\varepsilon)\cdot\left\|(x,y)\right\|_\alpha, \quad x,y\in E_1.$$

For $a_0=2$ and $\alpha-\beta<0$ we have

$$\frac{\varphi(a_i^n x, a_i^n y)}{a_i^{2n\beta}} = \frac{\delta(\varepsilon)}{a_i^{2n\beta}}\cdot\left\|(a_i^n x, a_i^n y)\right\|_\alpha \leqslant \delta(\varepsilon)\cdot a_i^{2n(\alpha-\beta)}\cdot\left\|(x,y)\right\|_\alpha \xrightarrow[n\to\infty]{} 0, \quad x,y\in E_1.$$

We obtain the same condition for $a_1 = \frac{1}{2}$ and $\alpha - \beta > 0$. Hence (5) is true.

Moreover, for $a_0 = 2$ and $\alpha - \beta < 0$ we get

$$\begin{aligned}\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \delta(\varepsilon) \cdot \left\|\left(\frac{x}{2}, \frac{x}{2}\right)\right\|_\alpha = \delta(\varepsilon) \cdot \left\|\left(2 \cdot \frac{x}{4}, 2 \cdot \frac{x}{4}\right)\right\|_\alpha \leqslant \\ \leqslant 4^\alpha \cdot \delta(\varepsilon) \cdot \left\|\left(\frac{x}{4}, \frac{x}{4}\right)\right\|_\alpha = 4^\alpha \cdot \varphi\left(\frac{x}{4}, \frac{x}{4}\right) = \\ = 4^\alpha \cdot \psi\left(\frac{x}{2}\right) = L \cdot 4^\beta \cdot \psi\left(\frac{x}{2}\right), \quad x \in E_1,\end{aligned}$$

where $L = 4^{\alpha-\beta} < 1$.

For $a_1 = \frac{1}{2}$ and $\alpha - \beta > 0$ we have

$$\begin{aligned}\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \delta(\varepsilon) \cdot \left\|\left(\frac{x}{2}, \frac{x}{2}\right)\right\|_\alpha \leqslant \frac{1}{4^\alpha} \cdot \delta(\varepsilon) \cdot \left\|(x, x)\right\|_\alpha = \\ = \frac{1}{4^\alpha} \cdot \varphi(x, x) = L \cdot \frac{1}{4^\beta} \cdot \psi(2x), \quad x \in E_1,\end{aligned}$$

where $L = 4^{\beta-\alpha} < 1$.

Therefore the inequality (4) is satisfied. So, in view of Theorem 3.1 there exists a unique quadratic mapping $Q\colon E_1 \to E_2$ such that

$$\left\|f(x) - f(0) - Q(x)\right\|_\beta \leqslant \frac{1}{4^\beta}\frac{1}{1-L}\psi(x), \quad x \in E_1$$

holds, with $L = 4^{\alpha-\beta}$, or

$$\left\|f(x) - f(0) - Q(x)\right\|_\beta \leqslant \frac{1}{4^\beta}\frac{L}{1-L}\psi(x), \quad x \in E_1$$

holds, with $L = 4^{\beta-\alpha}$.

Thus, the inequality (17) holds true for $\delta(\varepsilon) = \varepsilon \cdot 4^\alpha(4^\beta - 4^\alpha)$ and $\delta(\varepsilon) = \varepsilon \cdot 4^\alpha(4^\alpha - 4^\beta)$, respectively. □

Corollary 3.5. *Suppose that we have given a sub-homogeneous functional of order α on $E_1 \times E_1$, $\alpha \neq \beta$. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every odd function $f\colon E_1 \to E_2$ which satisfies*

$$\left\|f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y)\right\|_\beta \leqslant \delta(\varepsilon) \cdot \left\|(x, y)\right\|_\alpha, \quad x, y \in E_1,$$

there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\left\|f(x) - A(x)\right\|_\beta \leqslant \varepsilon \cdot \left\|(x, x)\right\|_\alpha, \quad x \in E_1. \tag{18}$$

Proof. Define

$$\varphi(x, y) := \delta(\varepsilon) \cdot \left\|(x, y)\right\|_\alpha, \quad x, y \in E_1.$$

For $a_0 = 2$ and $\alpha - \beta < 0$ we have

$$\frac{\varphi(a_i^n x, a_i^n y)}{a_i^{n\beta}} = \frac{\delta(\varepsilon)}{a_i^{n\beta}} \cdot \left\|(a_i^n x, a_i^n y)\right\|_\alpha \leqslant \delta(\varepsilon) \cdot a_i^{n(\alpha-\beta)} \cdot \left\|(x, y)\right\|_\alpha \xrightarrow[n\to\infty]{} 0, \quad x, y \in E_1.$$

We obtain the same condition for $a_1 = \frac{1}{2}$ and $\alpha - \beta > 0$. Hence (11) is true.

Moreover, for $a_0 = 2$ and $\alpha - \beta < 0$ we get

$$\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \delta(\varepsilon) \cdot \left\|\left(\frac{x}{2}, \frac{x}{2}\right)\right\|_\alpha = \delta(\varepsilon) \cdot \left\|\left(2 \cdot \frac{x}{4}, 2 \cdot \frac{x}{4}\right)\right\|_\alpha \leqslant$$
$$\leqslant 2^\alpha \cdot \delta(\varepsilon) \cdot \left\|\left(\frac{x}{4}, \frac{x}{4}\right)\right\|_\alpha = 2^\alpha \cdot \varphi\left(\frac{x}{4}, \frac{x}{4}\right) =$$
$$= 2^\alpha \cdot \psi\left(\frac{x}{2}\right) = L \cdot 2^\beta \cdot \psi\left(\frac{x}{2}\right), \quad x \in E_1,$$

where $L = 2^{\alpha-\beta} < 1$.

For $a_1 = \frac{1}{2}$ and $\alpha - \beta > 0$ we have

$$\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \delta(\varepsilon) \cdot \left\|\left(\frac{x}{2}, \frac{x}{2}\right)\right\|_\alpha \leqslant \frac{1}{2^\alpha} \cdot \delta(\varepsilon) \cdot \left\|(x, x)\right\|_\alpha =$$
$$= \frac{1}{2^\alpha} \cdot \varphi(x, x) = L \cdot \frac{1}{2^\beta} \cdot \psi(2x), \quad x \in E_1,$$

where $L = 2^{\beta-\alpha} < 1$.

Therefore the inequality (10) is satisfied. So, in view of Theorem 3.2 there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\left\|f(x) - A(x)\right\|_\beta \leqslant \frac{1}{2^\beta}\frac{1}{1-L}\psi(x), \quad x \in E_1$$

holds, with $L = 2^{\alpha-\beta}$, or

$$\left\|f(x) - A(x)\right\|_\beta \leqslant \frac{1}{2^\beta}\frac{L}{1-L}\psi(x), \quad x \in E_1,$$

holds, with $L = 2^{\beta-\alpha}$.

Thus, the inequality (18) holds true for $\delta(\varepsilon) = \varepsilon \cdot 2^\alpha(2^\beta - 2^\alpha)$ and $\delta(\varepsilon) = \varepsilon \cdot 2^\alpha(2^\alpha - 2^\beta)$, respectively. □

Corollary 3.6. *Let E_1 be a normed space over $\mathbb{K}$. Let us fix $p \in (0, 2)$, $\frac{p}{2} < \beta \leqslant 1$ in the case $i = 0$, and $p \in (2, \infty)$, $0 < \beta \leqslant 1$ in the case $i = 1$. If an even function $f\colon E_1 \to E_2$ satisfies the inequality*

$$\left\|f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y)\right\|_\beta \leqslant \varepsilon\big(\|x\|^p + \|y\|^p\big), \quad x, y \in E_1 \quad (19)$$

and $\varepsilon > 0$, then there exists a unique quadratic mapping $Q\colon E_1 \to E_2$ such that

$$\left\|f(x) - f(0) - Q(x)\right\|_\beta \leqslant \frac{2\varepsilon}{2^p \cdot |4^\beta - 2^p|}\|x\|^p, \quad x \in E_1. \quad (20)$$

Proof. Without loss of generality we can assume that $f(0) = 0$. Define $\varphi(x, y) := \varepsilon\big(\|x\|^p + \|y\|^p\big)$ for all $x, y \in E_1$. Moreover, let $L := L_i = \frac{a_i^p}{a_i^{2\beta}}$, where $i = 0$ when $p < 2\beta$, and $i = 1$ when $p > 2\beta$. Hence $0 < L < 1$ and

$$\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = 2\varepsilon \left\|\frac{x}{2}\right\|^p = 2\varepsilon \left\|a_i \cdot \frac{x}{2a_i}\right\|^p = 2\varepsilon a_i^p \left\|\frac{x}{2a_i}\right\|^p = \\ = a_i^p \varphi\left(\frac{x}{2a_i}, \frac{x}{2a_i}\right) = a_i^p \psi\left(\frac{x}{a_i}\right) = L \cdot a_i^{2\beta} \psi\left(\frac{x}{a_i}\right), \quad x \in E_1.$$

Furthermore,

$$\frac{\varphi(a_i^n x, a_i^n y)}{a_i^{2n\beta}} = \frac{\varepsilon\big(\|a_i^n x\|^p + \|a_i^n y\|^p\big)}{a_i^{2n\beta}} = \frac{\varepsilon \cdot a_i^{np}\big(\|x\|^p + \|y\|^p\big)}{a_i^{2n\beta}} = \\ = \left(\frac{a_i^p}{a_i^{2\beta}}\right)^n \cdot \varepsilon\big(\|x\|^p + \|y\|^p\big) = L^n \cdot \varepsilon\big(\|x\|^p + \|y\|^p\big) \xrightarrow[n\to\infty]{} 0, \quad x, y \in E_1.$$

Therefore, in view of Theorem 3.1 there exists a unique quadratic mapping $Q\colon E_1 \to E_2$ which satisfies (20). The proof is completed. □

Corollary 3.7. *Let E_1 be a normed space over $\mathbb{K}$. Let us fix $p \in (0,1)$, $p < \beta \leqslant 1$ in the case $i = 0$, and $p \in (1,\infty)$, $0 < \beta \leqslant 1$ in the case $i = 1$. If an odd function $f\colon E_1 \to E_2$ satisfies the inequality*

$$\big\|f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y)\big\|_\beta \leqslant \varepsilon\big(\|x\|^p + \|y\|^p\big), \quad x, y \in E_1 \quad (21)$$

and $\varepsilon > 0$, then there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\big\|f(x) - A(x)\big\|_\beta \leqslant \frac{2\varepsilon}{2^p \cdot |2^\beta - 2^p|}\|x\|^p, \quad x \in E_1. \quad (22)$$

Proof. Clearly, $f(0) = 0$. Define $\varphi(x,y) := \varepsilon\big(\|x\|^p + \|y\|^p\big)$ for all $x, y \in E_1$. Moreover, let $L := L_i = \frac{a_i^p}{a_i^\beta}$, where $i = 0$ when $p < \beta$, and $i = 1$ when $p > \beta$. Hence $0 < L < 1$ and

$$\psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = 2\varepsilon \left\|\frac{x}{2}\right\|^p = 2\varepsilon \left\|a_i \cdot \frac{x}{2a_i}\right\|^p = 2\varepsilon a_i^p \left\|\frac{x}{2a_i}\right\|^p = \\ = a_i^p \varphi\left(\frac{x}{2a_i}, \frac{x}{2a_i}\right) = a_i^p \psi\left(\frac{x}{a_i}\right) = L \cdot a_i^{\beta} \psi\left(\frac{x}{a_i}\right), \quad x \in E_1.$$

Furthermore,

$$\frac{\varphi(a_i^n x, a_i^n y)}{a_i^{n\beta}} = \frac{\varepsilon\big(\|a_i^n x\|^p + \|a_i^n y\|^p\big)}{a_i^{n\beta}} = \frac{\varepsilon \cdot a_i^{np}\big(\|x\|^p + \|y\|^p\big)}{a_i^{n\beta}} = \\ = \left(\frac{a_i^p}{a_i^{\beta}}\right)^n \cdot \varepsilon\big(\|x\|^p + \|y\|^p\big) = L^n \cdot \varepsilon\big(\|x\|^p + \|y\|^p\big) \xrightarrow[n\to\infty]{} 0, \quad x, y \in E_1.$$

Therefore, in view of Theorem 3.2 there exists a unique additive mapping $A\colon E_1 \to E_2$ which satisfies (22). This completes the proof. □

Corollary 3.8. *Let E_1 be a normed space over $\mathbb{K}$. Let $p \neq 1$, $p \neq 2$ and let us fix $\beta > p$ with $p \in (0,1)$, $\beta > \frac{p}{2}$ with $p \in (1,2)$ and $\beta > 0$ with $p \in (2,\infty)$, respectively.*

If a function $f\colon E_1 \to E_2$ *satisfies the inequality*

$$\big\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\big\|_\beta \leqslant \varepsilon\big(\|x\|^p+\|y\|^p\big), \quad x,y\in E_1$$

and $\varepsilon>0$*, then there exist a unique quadratic mapping* $Q\colon E_1\to E_2$ *and a unique additive mapping* $A\colon E_1\to E_2$ *such that*

$$\big\|f(x)-f(0)-Q(x)-A(x)\big\|_\beta \leqslant \frac{4\varepsilon}{2^{p+\beta}}\left(\frac{1}{|4^\beta-2^p|}+\frac{1}{|2^\beta-2^p|}\right)\|x\|^p, \quad x\in E_1. \tag{23}$$

Proof. Similarly as in the proof of Theorem 3.3 one can obtain the inequality (23). □

In the following two corollaries, we deal with the inequalities (19) and (21) for the case $p=0$. We need only to set $L=\frac{1}{4^\beta}$ and $L=\frac{1}{2^\beta}$ and apply Theorems 3.1 and 3.2 for their proofs, respectively. It is worth to note that the above constants L are the smallest ones satisfying conditions (4) and (10), respectively.

Corollary 3.9. *If an even function* $f\colon E_1\to E_2$ *satisfies the inequality*

$$\big\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\big\|_\beta \leqslant \varepsilon, \quad x,y\in E_1$$

for some $\varepsilon>0$*, then there exists a unique quadratic mapping* $Q\colon E_1\to E_2$ *such that*

$$\big\|f(x)-f(0)-Q(x)\big\|_\beta \leqslant \frac{\varepsilon}{4^\beta-1}, \quad x\in E_1.$$

Corollary 3.10. *If an odd function* $f\colon E_1\to E_2$ *satisfies the inequality*

$$\big\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\big\|_\beta \leqslant \varepsilon, \quad x,y\in E_1$$

for some $\varepsilon>0$*, then there exists a unique additive mapping* $A\colon E_1\to E_2$ *such that*

$$\big\|f(x)-A(x)\big\|_\beta \leqslant \frac{\varepsilon}{2^\beta-1}, \quad x\in E_1. \tag{24}$$

Corollary 3.11. *If a function* $f\colon E_1\to E_2$ *satisfies the inequality*

$$\big\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\big\|_\beta \leqslant \varepsilon, \quad x,y\in E_1$$

for some $\varepsilon>0$*, then there exist a unique quadratic mapping* $Q\colon E_1\to E_2$ *and a unique additive mapping* $A\colon E_1\to E_2$ *such that*

$$\big\|f(x)-f(0)-Q(x)-A(x)\big\|_\beta \leqslant \frac{2^{2-\beta}+2}{4^\beta-1}\varepsilon, \quad x\in E_1.$$

It is worth noticing that we can prove Theorem 3.2 with another definition of the number a_i. Namely, let us define a number a_i, $i=0,1$, by the formula:

$$a_i=\begin{cases} 3, & i=0,\\ \frac{1}{3}, & i=1.\end{cases}$$

Theorem 3.12. *Suppose $\varphi\colon E_1 \times E_1 \to [0,\infty)$ is a given function and there exists a constant L, $0 < L < 1$, such that the mapping*

$$x \to \psi(x) = \varphi(0,x), \quad x \in E_1$$

has the property

$$\psi(x) \leqslant L \cdot a_i^{\beta} \psi\left(\frac{x}{a_i}\right), \quad x \in E_1,\ i = 0,1,$$

and the mapping φ satisfies the condition

$$\lim_{n\to\infty} \frac{\varphi(a_i^n x, a_i^n y)}{a_i^{n\beta}} = 0, \quad x, y \in E_1,\ i = 0,1.$$

If an odd function $f\colon E_1 \to E_2$ satisfies the inequality

$$\big\| f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y) \big\|_{\beta} \leqslant \varphi(x,y), \quad x, y \in E_1,$$

then there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\big\| f(x) - A(x) \big\|_{\beta} \leqslant \frac{1}{6^{\beta}} \frac{L^i}{1-L} \psi(x), \quad x \in E_1,\ i = 0,1.$$

Corollary 3.13. *If an odd function $f\colon E_1 \to E_2$ satisfies the inequality*

$$\big\| f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y) \big\|_{\beta} \leqslant \varepsilon, \quad x, y \in E_1$$

for some $\varepsilon > 0$, then there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\big\| f(x) - A(x) \big\|_{\beta} \leqslant \frac{\varepsilon}{6^{\beta} - 2^{\beta}}, \quad x \in E_1.$$

It can be easily checked that the above approximation constant is better than that one obtained in (24).

Recently, B. Przebieracz [26] presented an application of the Markov-Kakutani common fixed point theorem to the theory of stability of functional equations by proving some version of the Hyers theorem concerning approximate homomorphisms. It seems to be interesting to consider applications of another fixed point theorems to the theory of the Hyers-Ulam stability of functional equations (see, e.g., fixed point theorems investigated in [7]).

Bibliography

1. Adam M.: *On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive functional equation.* Nonlinear Funct. Anal. Appl. **14**, no. 5 (2009), 699–705.
2. Aoki T.: *On the stability of the linear transformation in Banach spaces.* J. Math. Soc. Japan **2** (1950), 64–66.
3. Cădariu L., Radu V.: *Fixed points and the stability of Jensen's functional equation.* J. Inequal. Pure and Appl. Math. **4**, no. 1 (2003), art. 4.

4. Cădariu L., Radu V.: *On the stability of the Cauchy functional equation: a fixed point approach.* Grazer Math. Ber. **346** (2004), 43–52.
5. Cholewa P.W.: *Remarks on the stability of functional equations.* Aequationes Math. **27** (1984), 76–86.
6. Ciepliński K.: *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations – a survey.* Ann. Funct. Anal. **3** (2012), 151–164.
7. Czerwik S.: *Fixed Point Theorems and Special Solutions of Functional Equations.* Prace Nauk. Uniw. Śląsk. **428**, Katowice 1980.
8. Czerwik S.: *On the stability of the quadratic mapping in normed spaces.* Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
9. Czerwik S.: *The stability of the quadratic functional equation.* In: *Stability of Mappings of Hyers-Ulam Type*, Rassias Th.M., Tabor J. (eds.), Hadronic Press, Palm Harbor 1994, 81–91.
10. Czerwik S.: *Functional Equations and Inequalities in Several Variables.* World Scientific, New Jersey 2002.
11. Czerwik S.: *Stability of Functional Equations of Ulam-Hyers-Rassias Type.* Hadronic Press, Palm Harbor 2003.
12. Diaz J., Margolis B.: *A fixed point theorem of the alternative for contractions on a generalized complete metric space.* Bull. Amer. Math. Soc. **74** (1968), 305–309.
13. Fechner W.: *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings.* J. Math. Anal. Appl. **322** (2006), 774–786.
14. Forti G.L.: *An existence and stability theorem for a class of functional equations.* Stochastica **4** (1980), 23–30.
15. Forti G.L.: *Hyers-Ulam stability of functional equations in several variables.* Aequationes Math. **50** (1995), 143–190.
16. Gajda Z.: *On stability of additive mappings.* Internat. J. Math. & Math. Sci. **14** (1991), 431–434.
17. Gajda Z.: *Invariant Means and Representations of Semigroups in the Theory of Functional Equations.* Prace Nauk. Uniw. Śląsk. **1273**, Katowice 1992.
18. Hyers D.H.: *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
19. Hyers D.H., Isac G., Rassias Th.M.: *Stability of Functional Equations in Several Variables.* Birkhäuser Verlag, Basel 1998.
20. Hyers D.H., Rassias Th.M.: *Approximate homomorphisms.* Aequationes Math. **44** (1992), 125–153.
21. Jun K.-W., Kim H.-M., Lee D.O: *On the Hyers-Ulam-Rassias stability of a modified additive and quadratic functional equation.* J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. **11**, no. 4 (2004), 323–335.
22. Jung S.-M.: *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis.* Hadronic Press, Palm Harbor 2000.
23. Kirk W.A., Sims B. (eds.): *Handbook of Metric Fixed Point Theory.* Kluwer Academic Publishers, Dordrecht 2001.
24. Kuczma M.: *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality.* Second edition (edited by A. Gilányi), Birkhäuser Verlag, Basel 2009.
25. Páles Z.: *Generalized stability of the Cauchy functional equation.* Aequationes Math. **56** (1998), 222–232.
26. Przebieracz B.: *The Hyers theorem via the Markov-Kakutani fixed point theorem.* J. Fixed Point Theory Appl. **12** (2012), 35–39.
27. Rassias Th.M.: *On the stability of the linear mapping in Banach spaces.* Proc. Amer. Math. Soc. **72** (1978), 297–300.
28. Rätz J.: *On approximately additive mappings.* In: *General Inequalities 2*, Beckenbach E.F. (ed.), Birkhäuser Verlag, Basel 1980, 233–251.
29. Sikorska J.: *On a direct method for proving the Hyers-Ulam stability of functional equations.* J. Math. Anal. Appl. **372** (2010), 99–109.
30. Skof F.: *Proprietá locali e approssimazione di operatori.* Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
31. Szekélyhidi L.: *Note on a stability theorem.* Canad. Math. Bull. **25** (1982), 500–501.
32. Tabor J., Tabor J.: *General stability of functional equations of linear type.* J. Math. Anal. Appl. **328** (2007), 192–200.
33. Ulam S.M.: *Problems in Modern Mathematics.* Wiley, New York 1960.